A utility function based approach towards the modeling of migration in village equilibrium models

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Abstract

Village equilibrium models are computable general equilibrium (CGE) implementations of agricultural household models in a village equilibrium framework which have the salient feature of being able to capture general equilibrium effects arising at the level of rural communities. Due to the important role migration plays for livelihoods in developing countries, the approach has been successfully applied to analyze aspects related to migration and village economies. However, the depiction of migration in village equilibrium models is not carried out in a way that captures interactions between migration and household consumption demand while at the same time allows for an endogenous adjustment of the level of migration by the households themselves. Furthermore, approaches to modeling migration are purely demand side oriented. Supply side factors, such as

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differences between households, which may influence household responses to changes in incentives to migrate, cannot be accommodated in a theoretically convincing manner. To address these issues, a nonseparable household model with endogenous migration decisions and feedback to the consumption sphere is proposed as the theoretical foundation for a village equilibrium model. A composite utility function captures utility which accrues to the household through per capita household consumption of goods and leisure, on the one hand, and utility stemming directly from participation in different activities by the household including migration, on the other hand. It is shown that the allocation of labor among different activities is governed by the size of marginal returns to labor in terms of market returns, changes in household demand and (dis)utility of labor market participation relative to the household shadow wage. The practical implementation of the theoretical framework is achieved by the derivation of two independent demand systems from the composite utility function. A per capita linear expenditure system is proposed to depict household consumption demand. The allocation of labor to migration is assumed to follow a factor demand specification using power functions which translate utility considerations made by the household into imperfectly elastic responses to changes in incentives for participation in the labor market.

Keywords: Migration, village equilibrium modeling, general equilibrium modeling, computable general equilibrium, village CGE, agricultural household model

1 Introduction

Village equilibrium models are computable general equilibrium (CGE) implementations of agricultural household models in a village equilibrium framework which are used to describe and depict village economies in developing countries. As an overlap of CGE modeling and microsimulation, village equilibrium models have the salient feature of being able to take into account heterogeneity of economic actors up to a substantial degree as well as to capture general equilibrium effects arising at the rural community level. The provision for heterogeneous agents reduces the aggregation bias inherent to macroeconomic modeling approaches. At the same time, the consideration of interactions among agents, which take place at a local level, catches important characteristics of economic systems which are neglected in microeconomic agricultural household models.
Thus, village equilibrium models constitute useful tools to analyze policy outcomes for small groups with varying characteristics within a population, while these models accurately capture the transmission of a particular economic shock throughout a local economy.

In the past, village equilibrium models were used for addressing different aspects related to labor migration. More recently, the topic of national and international migration has assumed a prominent position in the international development debate (see, for example, UNDP, 2009). Both the early appearance of migration issues in village equilibrium studies and the recent emergence of the topic in the political arena are because migration and remittances in many occasions not only make up substantial shares of household income, but also constitute important means of coping with adverse shocks which threaten the viability of a household’s economy. Because of the latter, for an economic assessment of the role and relevance of migration, it is not only the contribution of migration and remittances to total household income, but also the migration response of a household following changes in the economic environment which matters. The migration response, in turn, is determined by a range of socio-economic characteristics of the household which could be summarized as supply side characteristics. In this context, this paper proposes an alternative approach to modeling migration in village equilibrium models. The goal is to achieve an accurate depiction of the migration behavior of rural households which allows the model to take into account a household’s supply side characteristics in a theoretically consistent manner.

Representing a work in progress, this paper begins with a short review of the literature on village equilibrium modeling with a special emphasis on the depiction of migration. In Section 3 an agricultural household model is developed. A composite utility function allowing for a supply side oriented modeling of migration is the central piece of this model. The household model constitute the theoretical core of a stylized village equilibrium model, which is presented in Section 4. The paper concludes with a section which elaborates on a possible implementation of the model, arguing that the proposed approach can be applied to a wider range of issues involving, for example, labor allocation in regionalized general equilibrium models.

\[1\] And, of course, remittances constitute substantial shares the GDP of entire national economies.
2 Literature review

Village equilibrium modeling

Village equilibrium models are built upon micro-level social accounting matrices (SAM) which, by providing a consistent snapshot of a village economy at a certain point in time, have soon become the preferred framework for carrying out further analyses. Departing from village SAMs, early village level modeling studies applied SAM multiplier approaches, including multiplier decomposition (Pyatt and Round, 1979) and structural path analysis (Defourny and Thorbecke, 1984), with the aim of exploring the nature and strength of economic linkages within as well as assessing the impact of economic shocks on local economies. Issues which have been studied include, for example, the effect of changes in inflows of remittances and government transfers (Adelman, Taylor, and Vogel, 1988), impacts of output fluctuations and investment in irrigation (Subramanian and Sadoulet, 1990) or the assessment of alternative rural development schemes (Parikh and Thorbecke, 1996).\footnote{Although not strictly at the village but rather at a regional level, a study by Lewis and Thorbecke (1992) which analyzes aggregate and household level impacts of sectoral changes in production should be mentioned in this context, as well. Furthermore, more recent studies which apply multiplier approaches to village level data exist (Yuñez Naude, Dyer, and Taylor, 2006; Subramanian and Qaim, 2009).}

Recognizing the rather restrictive assumptions of the SAM multiplier approach, Taylor and Adelman (1996) developed a first village CGE model by embedding a neoclassical agricultural household model (Singh, Squire, and Strauss, 1986) into a local general equilibrium framework. Compared to SAM multiplier models, the village equilibrium model has the advantage of abandoning the fixed price assumption as well as the advantage of allowing for a much more flexible depiction of the behavior of economic agents. Moreover, the Taylor-Adelman model incorporates the assumption of nonseparable household decisions, an important feature of agricultural household models which helps to explain behavioral patterns which otherwise might appear irrational from an economic perspective (de Janvry, Fafchamps, and Sadoulet, 1991; de Janvry and Sadoulet, 2003). The main insight from nonseparable household models is that decisions of a household subject to nonseparability are not governed by market prices, which are exogenous to the household alone, but are instead governed by endogenous shadow prices determined inside the household. Nonseparability implies that household behavior can no longer be analyzed in a separable and recursive
manner by first optimizing income from household production and then utility from consumption (Singh, Squire, and Strauss, 1986). It is rather necessary to consider maximization of profit and utility as interdependent optimization processes. As this interdependence affects the comparative statics of a household model (Singh, Squire, and Strauss, 1986; Lopez, 1986), its implementation in a village equilibrium model constituted a major step towards a more realistic depiction of household behavior.

Following Singh, Squire, and Strauss (1986), Taylor and Adelman (1996) assume missing markets for family labor and land as the reason underlying the nonseparable nature of household decisions in their model; however, household decisions may become nonseparable due to a larger variety of conditions. Nonseparability may occur when farm households act in an imperfect market environment (de Janvry, Fafchamps, and Sadoulet, 1991; de Janvry and Sadoulet, 2003; Singh, Squire, and Strauss, 1986). Reasons why markets might be imperfect or even fail include variable transaction costs on product or factor markets, fixed transaction costs which constitute market entry barriers or constraints on market participation, and missing markets such as for capital, land or labor (see, for example, de Janvry and Sadoulet, 2003). Apart from market imperfections, nonseparability may be caused by imperfect substitutability between hired labor and family labor or by preferences of households regarding the participation in certain employment activities (Lopez, 1986; Singh, Squire, and Strauss, 1986; Skoufias, 1994). Nevertheless, these reasons for nonseparability have not been considered yet in village equilibrium models.

Kuiper (2005), however, recognizes the rather strong character of the assumption on the reason which underlie the nonseparability of household behavior in the Taylor-Adelman model. Offering an extension of the village equilibrium approach, she introduces fixed transaction costs in product and factor markets into her village equilibrium study of a rural community in China. This implementation of the price band model proposed by Sadoulet and de Janvry (1995) relaxes the strong assumption of missing factor markets and allows for the con-

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3There are a number of studies available which carry out analyses based on a nonseparable agricultural household model under different assumptions regarding the reasons which lead to nonseparability. Benjamin (1992), Sadoulet, de Janvry, and Benjamin (1998), Sonoda and Maruyama (1999) and Sonoda (2008), for example, assume imperfect labor markets. Carter and Yao (2002) carry out an analysis of nonseparability due to imperfections in the market for land. Benjamin (1992) adds the case of nonseparability caused by imperfect substitutability of hired and family labor and Lopez (1984, 1986) presents a nonseparable model in which households have preferences regarding different occupations.
sideration of imperfect markets as an intermediate case between a perfect and a missing market. To date, the Kuiper model represents the latest development in village equilibrium modeling.  

Migration in village equilibrium models

The modeling of migration in village equilibrium models exploits the possibility to flexibly incorporate assumptions on household behavior. This offers scope for a realistic depiction of the migration behavior of households and their migration responses due to economic shocks. Moreover, village equilibrium models are able to capture potential impacts of economic shocks which can be, for example, an assumed variation in migration or flows of remittances or any change in economic policies which in turn may provoke alterations in migration and remittances on all members of a local community, including those who are not directly involved in migration. Accordingly, the approach has been successfully applied to the study of different aspects of migration. Taylor, Yuñez Naude, and Dyer (1999) and Taylor, Yuñez Naude, and Hampton (1999), for instance, analyze the impacts of alternative agricultural and trade policy scenarios on production, income and migration in rural Mexico.  

Kuiper (2005) simulates the effects of an increase in migration on production and consumption in a Chinese village. In a similar fashion, Kuiper and van Tongeren (2006) administer a migration shock which is part of a broader Doha Round trade liberalization scenario to the same Chinese village model. All studies highlight the importance not only of migration, but also of economic interactions within a village and local general equilibrium effects for the nature of a particular policy outcome.

With respect to the modeling of migration, the village equilibrium studies cited above use two different approaches. Taylor, Yuñez Naude, and Dyer (1999) and Taylor, Yuñez Naude, and Hampton (1999) apply the model developed by Taylor and Adelman (1996). In this model, the level of migration is determined endogenously, as households allocate labor to migration until the marginal returns to migration (i.e. remittances) equal the marginal returns from labor in each alternative income generating activity. The marginal returns, in turn, correspond to the household shadow wage. The household shadow wage itself

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4 Applications of this model include Heerink, Kuiper, and Shi (2006), Heerink et al. (2007) and Kuiper and van Tongeren (2006).

5 Further studies, which apply the approach economy-wide are Materer and Taylor (2003) and Taylor and Dyer (2009).
reflects the marginal valuation of family time and leisure. That is, the extent to which a household engages in migration is determined to be an equilibrium between the allocation of family time to migration and other activities and the consumption of leisure. While this approach offers the great advantage of allowing for endogenous changes in the level of migration as a response to a given economic shock, it captures only two ways of how migration impacts the household economy; namely the income from migration which accrues to the household and the competition between migration, other activities and leisure for the scarce time the household is endowed with (Taylor and Adelman, 1996, p.185). It neglects, however, the impact of migration on consumption demand due to migration related changes in the household size which can constitute an additional benefit to the household.\(^6\) Furthermore, the approach does not take into account potential disutility generated through the engagement in migration. This, as the authors themselves acknowledge, may lead to an overestimation of the level of migration (Taylor and Adelman, 1996, p.185). Kuiper (2005) tackles the issue of the consumption impacts of migration by implementing a per capita demand system, thus creating a feedback between the level of migration and the consumption sphere of the household.\(^7\) Still, a change in migration is modeled as an exogenous shock and not as a decision endogenous to the household.

Following this discussion, two major challenges with respect to modeling migration in village equilibrium models can be identified. The first challenge consists of modeling migration as a decision endogenous to the household while simultaneously taking into account the impact of migration on consumption demand. The second challenge, related to the question of disutilities arising from participation in migration, involves a more general issue with respect to the depiction of the migration responses of households. In principle, both Taylor and Adelman (1996) and Kuiper (2005) model migration as demand driven in the sense that an external demand shock triggers a supply response by households. Such an external demand shock can either consist of changes of the wage rate in migration, i.e. in the returns to migration as in case of Taylor and Adelman (1996), or of a change in employment in migration as in the studies by Kuiper. In reality, however, it can be observed that households respond differently to changes in incentives to migration and these differences are often due to supply

\(^6\)It constitutes a benefit in case of increasing household migration as there will be less persons with demand for consumption. In case of decreasing migration, total demand inside the household will increase. Hence, the competition for income intensifies.

\(^7\)In fact, the author applies a per capita linear expenditure system (LES).
side characteristics of the households. The presence of children or elderly, for example, may make migration a less attractive option for a young couple. Likewise, the need for childcare may require at least one person to stay at home (i.e. on the farm) and make farm work the preferred option for this person. It can be argued that in both of the current village equilibrium modeling approaches it is possible to accommodate migration responses which differ among households. In the Kuiper model one could simply define household group specific migration shocks according to assumed differences in migration responses. In the Taylor-Adelman model differences in migration responses can be implemented through the household specific calibration of an elasticity of remittances with respect to family time allocated to migration which forms part of a remittances function.\footnote{Taylor and Adelman (1996) do exactly this but state differential access to migrant labor markets rather than supply side factors as the underlying rationale.} Nonetheless, both approaches would be rather \textit{ad hoc} and would lack a sound theoretical base with respect to the supply side considerations mentioned above.

The village equilibrium model developed in this paper addresses each of the challenges identified. The village model is built around an agricultural household model which features a composite utility function consisting of a per capita Stone-Geary utility function for consumption and a sum of power functions for (dis)utility generated through the participation in different income generating activities. The first part of the composite utility function ensures that feedbacks to the consumption sphere are taken into account. The second part of the function provides for the incorporation of supply side related differences in migration responses between households which arise due to differences in socio-economic characteristics. The following section presents the theoretical model of household behavior in a general form and derives the comparative statics of the model. After the specification of functional forms, a proposal for the implementation of the theoretical model in a stylized village equilibrium model is provided in Section 4.

\section{A theoretical model of household behavior}

The theoretical model presented in this section is set up in the spirit of the agricultural household model developed and first presented by Singh, Squire, and Strauss (1986). Apart from a missing market for family labor, the agricultural household is assumed to operate in an environment of perfect markets in the
neoclassical sense. In other words, it is assumed that there are no transaction costs, no barriers to market entry and the household disposes of perfect information. The household consumes an agricultural good $X_a$, a manufactured good $X_p$ and leisure $X_l$. The agricultural good is produced by the household and sold, while the manufactured good has to be purchased from the market. The productive activities the household can engage in comprise agriculture, which produces the agricultural good, local off-farm employment and migration. The household maximizes utility with respect to the goods consumed, the labor time allocated to agriculture $T_f$, to local off-farm employment $T_o$ and to migration $T_m$, as well as a variable capital input $V$ and land $A$ which are both used in agricultural production.

The core piece of the treatment of time allocation, including migration of the household in the model, is a composite utility function. The first component of this function is a consumption utility function $U^C$ which captures utility generated through the consumption of the three consumption goods. The consumption utility function allows for feedbacks from migration to the consumption sphere through its formulation on a per capita basis. This component basically draws on work by Wouterse (2006) who analyzes migration of rural households in Burkina Faso. Choices of labor market participation are included into the preference structure of the household through a second component of the composite utility function, denoted $U^L$. $U^L$ takes into account utility which stems from the time allocated to farming, off-farm employment and migration. This follows an approach pioneered by Lopez (1984, 1986) and further applied by Sonoda (2008).

Formally, the composite utility function and the associated utility maximization problem of the household can be defined as

$$\max_{X_a,X_p,X_l,T_f,T_o,T_m,V,A} U(X_a,X_p,X_l) + U^L(T_f,T_o,T_m)$$  \hspace{1cm} (3.1)

As mentioned above, the quantities included in $U^C$ are on a per capita basis.\(^9\)

Utility is maximized subject to a production constraint which is represented through an agricultural production function

$$Q_a = f(T_f, V, A)$$  \hspace{1cm} (3.2)

\(^9\)Actually, $U^C$ is defined per adult equivalent and is referred to throughout the paper when using the term "per capita".
where $Q_a$ is the quantity of agricultural output produced and $T_f$, $V$ and $A$ are the inputs used in agricultural production. A time constraint ensures that the total time use of the household for production and consumption equals its time endowment $\bar{T}$:

$$\bar{T} = T_f + T_o + T_m + X_l (N - M)$$

Please note that in the time constraint $X_l$, which is defined on a per capita basis, is scaled to total leisure consumption of the household by multiplication with a term $(N - M)$. This term describes the number of household members in the economically active age who actually live in the household and express demand for consumption, equaling the difference between the total number of persons in the economically active age in the household and the number of migrants $M$. $M$, the relevant number of migrants, in turn, is calculated as $\frac{T_m}{E}$ with $E$ being the time period covered by the analysis and $T_m$ being the time worked in migration, both expressed in the same unit of measurement. In addition to the time constraint, a land constraint states that the area of land $A$ used by the household must equal the household’s endowment with land $\bar{A}$ plus rentals $R$ from a land rental market:

$$A = \bar{A} + R$$

Depending on whether the demand for land from the agricultural production activity exceeds or falls below the land endowment, $R$ becomes positive or negative.

Next, a function of remittances accounts for the fact that migrants do not transfer the entire income they earn to their families, but rather a fraction of it. The remittances function incorporates the assumption that remittances are a function of the time dedicated to migration and the wage rate $w_m$ which prevails at the destination.\footnote{This is different from the remittances function used by Taylor and Adelman (1996) who do not explicitly include the wage rate at the destination.}

$$REM = r(T_m; w_m)$$

\footnote{For example, if the time period under consideration is one year, the absence of a migrant during half a year would be equivalent to 0.5 less economically active persons (which otherwise would demand leisure and consumption goods) in the household.}
Finally, a cash income constraint includes the requirement that total expenditures of the household must equal total income $I^T$.

$$p_a Q_a + w_o T_o + REM + a R = (N - \frac{T_m}{E} + \Delta D) (p_a X_a + p_p X_p) + v V \equiv I^T \quad (3.6)$$

Similar to the time constraint, consumption of $X_a$ and $X_p$ is scaled to total amounts consumed by the household through multiplication with a term $(N - \frac{T_m}{E} + \Delta D)$, in which $D$ is the number of dependents living in the household and $\Delta$ is a parameter which scales $D$ to adult equivalent consumption levels. $p_a$ and $p_p$ represent the prices of the agricultural good and the consumption good, respectively, $w_o$ is the prevailing wage rate in local off-farm employment, $a$ the land rental rate and $v$ the price of the variable input.

The production function (3.2), the land constraint (3.4), the remittances function (3.5) and the cash income constraint (3.6) are collapsed into a single combined constraint

$$p_a f(T_f, V, A) + w_o T_o + r(T_m; w_m) + a(A - A) = (N - \frac{T_m}{E} + \Delta D) (p_a X_a + p_p X_p) + v V \quad (3.7)$$

The maximization of the composite utility function (3.1) subject to the time constraint (3.3) and the combined constraint (3.7) leads to the conditions governing consumption demand, time allocation and factor demand of the household.

The part of the model of highest interest from the perspective of this paper, of course, is the way in which the household divides its time between leisure and income generating activities, including migration. The first order conditions which result from partial differentiation of the Lagrangian expression associated with the household problem (see Annex A) can be manipulated to obtain the condition which governs household migration.

$$\frac{1}{\lambda} \frac{\partial U^L}{\partial T_m} + \frac{\partial r(\cdot)}{\partial T_m} + (p_a X_a + p_p X_p + \frac{\psi}{\lambda} X_l) = \frac{\psi}{\lambda} \quad (3.8)$$

Equation (3.8) shows that to maximize utility, the household allocates labor to migration up to the point where the returns from migration equal the household shadow wage $\frac{\psi}{\lambda}$ (i.e. the marginal utility of time $\psi$ translated into value terms is divided by the marginal utility of income $\lambda$). The returns from migration,
in turn, consist not only of remittances, but also of gains in value terms from lower demand for consumption and leisure plus a change in utility due to the engagement in the migration activity.\textsuperscript{12}

By extending the analysis to include the allocation of labor by the household to all productive activities as well as leisure, one obtains a complete picture of the household’s labor allocation:

\[
\frac{\psi}{\lambda} = \frac{1}{\lambda} \frac{\partial U^C}{\partial X_l(N - \frac{T_m}{\psi})} = \frac{1}{\lambda} \frac{\partial U^L}{\partial T_f} + \frac{p_a}{\lambda} \frac{\partial f(\cdot)}{\partial T_f} = \frac{1}{\lambda} \frac{\partial U^L}{\partial T_o} + w_o = \frac{1}{\lambda} \frac{\partial U^L}{\partial T_m} + \frac{\partial r(\cdot)}{\partial T_m} + (p_a X_a + p_p X_p + \frac{\psi}{\lambda} X_l)
\]

According to Equation (3.9), at the household’s optimum all activities and the consumption of leisure yield the same marginal returns to the household. These marginal returns include the monetary returns as well as a utility component and are equal to the household shadow wage $\frac{\psi}{\lambda}$. An implication of this result is that household decisions become nonseparable. Indeed, there is interdependence between the allocation of labor to productive activities and the consumption of leisure, mediated by the shadow wage which differs from any of the market wage rates and is endogenous to the household. In contrast to previous approaches to village modeling, nonseparability is not caused by market imperfections or missing markets. Instead, it is caused by the preferences of the household regarding the participation in different activities. While resembling the result by Lopez (1984, 1986), this constitutes a novelty in the village equilibrium literature.

A further implication of Equation (3.9) worth pointing out is that the model allows the household to be engaged in various income generating activities simultaneously and wage rates are allowed to differ. Up to the present, it has been necessary to assume profits on off-farm activities combined with restrictions on the amount of labor a household can allocate to a particular activity to avoid complete specialization in a situation of differing wage rates (compare

\textsuperscript{12}Originally, the third term on the right hand side of the equation is $\frac{1}{\lambda} (p_a X_a + p_p X_p + \frac{\psi}{\lambda} X_l)$; however, the possibly unusually looking term $\frac{1}{\lambda}$ becomes 1 when the time period under consideration is assumed to be 1 (year or any other time period). This simplifies this as well as following expressions a bit.
As illustrated below, with the model presented here it is possible to abandon this assumption and dispense with quantitative restrictions on particular activities.

It should also be noted that under the rather unrestrictive assumptions regarding the utility function $U^L$ it remains 	extit{a priori} undetermined how the household’s shadow wage relates to the market’s wage rates. In case utility is an increasing function of the time spent in an activity (i.e. the household experiences pleasure from being engaged in an activity which goes beyond the mere wage income), the shadow wage may be higher than the respective market wage. In case utility decreases with increasing time worked, the household shadow wage is lower than the market wage.\textsuperscript{13} The actual decision for which assumption is more valid, meanwhile, is an empirical matter.

Corresponding to the requirement formulated above, the model gives rise to the option to model migration responses which differ among households. Unlike other current approaches to village modeling, the explicit inclusion of the utility connotation of migration provides a theoretical concept which includes the supply side considerations of the household, thus departing from a pure demand driven modeling of migration. Depending on considerations related to their socio-economic characteristics, households may exhibit stronger or weaker responses to changes in incentives to migrate, e.g. to changes in relative wage rates.

The conditions governing the demand for agricultural and manufactured consumption goods by the household largely correspond to standard demand conditions as obtained from microeconomic demand theory. The difference is that the conditions reflect the formulation of the consumption utility function $U^C$ on a per-capita basis.

$$\frac{\partial U^C}{\partial X_a(N - \frac{T_m}{E} + \Delta D)} = \lambda p_a ; \quad \frac{\partial U^C}{\partial X_p(N - \frac{T_m}{E} + \Delta D)} = \lambda p_p$$ \hspace{1cm} (3.10)

According to Equation (3.10), the household consumes a product up to the point where the marginal utility of consumption of the good is equal to the marginal utility of the income spent on the marginal unit consumed. Unlike a standard demand function, the amount of the good is scaled to the household

\textsuperscript{13}This is the assumption made by Lopez (1984, 1986) and Sonoda (2008). Furthermore, this assumption will be adopted in the village equilibrium model presented in the following section.
level through multiplication with a term which takes into account the change in household size due to migration. Household demand for leisure has already been included in Equation (3.9).

The results for demand for land and variable inputs for agricultural production is straightforward. Demand by the agricultural activity for land obeys

$$ p_a \frac{\partial f(\cdot)}{\partial A} = a $$

and the demand for the variable input follows

$$ p_a \frac{\partial f(\cdot)}{\partial V} = v. $$

Both equations imply that land and the variable input are demanded up to the point where their marginal value products equal the price of the respective factor. This, again, is a standard result from microeconomic production theory.

The results for factor demand constitute the last component of the nonseparable agricultural household model which forms the core piece of the village equilibrium model to be developed in the following section. With the aim of illustrating a potential method for how the developed theory can be applied and how its theoretical features can be exploited, the next section proceeds with translating the theoretical features into an applied model and, as a result, presents a stylized village equilibrium model.

## 4 A stylized village equilibrium model

To translate the theoretical household model into a village CGE format which can be used to carry out simulation analyses, it is necessary to specify functional forms for the utility functions, remittances function and agricultural production function. Furthermore, assumptions have to be made regarding the tradability of commodities and factors. Finally, a village equilibrium framework which accommodates these assumptions needs to be constructed. The activities of the households which constitute the village community take place within this framework.

As became clear in Section 3, the allocation of labor to migration, other productive activities and leisure is largely determined by utility considerations of
the households: The households derive utility from the consumption of leisure and their level of utility is affected by participation in different productive activities. Accordingly, the composite utility function proposed consists of a part reflecting consumption utility as well as a part generating utility from labor market participation. Due to the necessity of capturing the effects of changes in household consumption demand which arise following changes in migration, the consumption utility function $U^C$ is specified as a per capita expenditure system. As a compromise between keeping matters simple and achieving a realistic depiction of household behavior, a per adult equivalent Stone-Geary utility function is chosen to represent consumption demand (see Kuiper, 2005, for a former application of this approach). The labor market participation component $U^L$ of the composite utility function in this illustrative application is assumed to exhibit negative marginal utility of labor allocated to the different activities. Furthermore, it is assumed that the absolute value of the marginal utility increases with the amount of labor allocated to a particular activity. This implies that households experience a certain degree of disutility from participation in any income generating activity which increases with the amount of labor. A simple sum of power functions is proposed here. To avoid undue complexity of the model, remittances and agricultural production are dealt with using rather simple functional forms. Remittances are assumed to be a linear function of the product of time allocated to migration and the wage rate. A Cobb-Douglas production technology is used to model agricultural production.

Assumptions regarding the tradability of products and factors determine the mechanisms defining prices and lead to the general equilibrium framework of the model. The six products and factors contained in the simple model developed in this paper are tradable at different levels. The agricultural good $X_a$, the manufactured good $X_p$ and the variable input for agricultural production are assumed to be traded outside the village. Following Taylor and Adelman (1996) these goods, which can be exported and imported at a price given by the outside world, are denoted village tradables. Land is assumed to be traded inside the village. This gives rise to a village rental market in which the rental rate for land is determined by supply and demand within the village. Goods or factors traded among households within the village and not with the outside world are referred to as household tradables. Family labor, finally, takes a special position. While labor is also traded outside the village, different wage rates including the household shadow wages apply. The wage rate in off-farm activ-
ities is fixed outside the village. The household shadow wages, however, which ultimately governs the time allocation of the households is determined within the households. Furthermore, the assumption that labor cannot be purchased by households plays a role in defining the households’ balance of family labor (see below).

Before beginning the depiction of the village equilibrium model, some aspects regarding notation should be mentioned. First, an index $h$ is added to household specific variables to represent the different households which constitute the village community. Each $h$ can be considered as an element of the set $H = \{1, \ldots, n\}$. Furthermore, two additional sets, $G = \{a, p, l\}$, which contains all consumption goods, and $B = \{f, a, m\}$, which incorporates the different productive activities, are defined.

Following the considerations made above, the composite utility function is

$$U_h = U^C_h + U^L_h = \prod_{g \in G} (X_{gh} - \sigma_{gh})^\gamma_{gh} + \sum_{b \in B} (-\varepsilon_{bh}T_{bh}) \quad \forall h \in H.$$ (4.1)

The first term on the right hand side (the second line) is the consumption utility function $U^C$ in which all $X_g$ are defined per adult equivalent. $\sigma_{gh}$ describes the fixed committed (or subsistence) consumption quantities and $\gamma_{gh}$ are the marginal expenditure shares. The second term constitutes the utility function for labor market participation $U^L$. The parameters $\varepsilon_{bh}$ and $\delta_{bh}$ determine how time allocated to a particular activity translates into utility. The negative sign which precedes $T_{bh}$ ensures that the households experience a disutility from labor market participation.

Constrained maximization of the utility function (4.1) with respect to consumption goods and leisure leads to a per capita linear expenditure system (LES) (Sadoulet and de Janvry 1995, p.42; Kuiper 2005, p.137ff.). Demand for the agricultural and manufactured good is described by

$$X_{ah} = \frac{p_a \sigma_{ah} + \gamma_{ah}(I_h - \sum_{g \in G} p_g \sigma_{gh})}{p_a} \quad \forall h \in H.$$ (4.2)

and

$$X_{ph} = \frac{p_p \sigma_{ph} + \gamma_{ph}(I_h - \sum_{g \in G} p_g \sigma_{gh})}{p_p} \quad \forall h \in H.$$ (4.3)
where $p_{lh}$, the price for leisure is $\psi_{lh}$ (see below) which – unlike prices of the other consumption goods – differs among households.\textsuperscript{14} Demand for leisure is

\begin{equation}
X_{lh} = \frac{\psi_{lh}}{\bar{\lambda} h} \sigma_{lh} + \gamma_{lh} (I_h - \sum_{g \in G} p_{gh} \sigma_{gh}) \quad \forall h \in H \tag{4.4}
\end{equation}

In equations (4.2) to (4.4), quantities and income are defined per adult equivalent. For remaining parts of the model total quantities per household are used. Thus, it is necessary to establish a relationship which scales per adult equivalents to total quantities:

\begin{equation}
X_{gh}^T = X_{gh} \left( N_h - \frac{T_{mh}}{E} + \Delta D_h \right) \quad \forall g \in G, h \in H. \tag{4.5}
\end{equation}

Likewise, per adult equivalent income is scaled to total household income by

\begin{equation}
I_{gh}^T = I_{gh} \left( N_h - \frac{T_{mh}}{E} + \Delta D_h \right) \quad \forall h \in H. \tag{4.6}
\end{equation}

Equations (4.2) to (4.5) together constitute the expenditure block of the village equilibrium model which comprises commodity and leisure demand. As a result of the expenditure system being defined in a per adult equivalent basis, the time dedicated to migration (i.e. the time spent by a migrant outside the household) exerts a direct influence on consumption demand in Equation (4.6). This establishes the feedback between migration and household consumption. Household income, in turn, has not been defined yet. It consists of the returns to the factors owned by the households and employed in the three different activities. Hence, before defining total household income, the production functions for each activity will be stated along with the associated factor demand functions, i.e. the allocation of factors to respective activities. As migration is in the focus of this paper, we begin with the treatment of this activity.

As already indicated above, any kind of off-farm employment is treated as a production activity. The output of this activity is a commodity sold to the labor market. Consequently, the simple production function directly maps the amount of labor supplied into an output of the activity.\textsuperscript{15} In case of migration,
the production function is

\[ Q_{mh} = T_{mh} \quad \forall \, h \in H \]  

(4.7)

where \( Q_{mh} \) is the amount of labor sold to a migrant labor market. In addition, the treatment of the allocation of labor to migration requires the specification of a remittances function. While more complicated functional forms are conceivable, a simple linear function is chosen:

\[ REM_h = \kappa_h Q_{mh}w_m \quad \forall \, h \in H. \]  

(4.8)

Apart from the time allocated to migration and the wage rate which prevails at the destination, the equation contains a parameter \( \kappa_h \) which takes a value between 0 and 1, determining the share of a migrant’s income which is accrued to the households in the form of remittances. Following Equation (3.9), factor demand for the migration activity or, equivalently, the allocation of household labor to migration has to fulfil the condition

\[
\kappa_h w_m = \frac{\psi}{\lambda_h} + \frac{1}{\lambda_h} \varepsilon_{T_{mh}} \delta_{T_{mh}} T_{mh}^{\delta T_{mh} - 1} - \left( p_a X_{ah} + p_p X_{ph} + \frac{\psi}{\lambda_h} X_{lh} \right) \quad \forall \, h \in H
\]  

(4.9)

which states, as above, that in the optimum the households equate marginal returns from migration with marginal costs. The marginal costs, in turn, consist of three components. The first component is the household specific shadow wage \( \frac{\psi}{\lambda_h} \), i.e. the opportunity cost of the factor. The second component \( \frac{1}{\lambda_h} \varepsilon_{T_{mh}} \delta_{T_{mh}} T_{mh}^{\delta T_{mh} - 1} \), which will be discussed in detail below, reflects the disutility arising from migration to the specific household. This generates a markup to the shadow wage and, equivalently, diminishes the value of the returns from migration. The third component \( (p_a X_{ah} + p_p X_{ph} + \frac{\psi}{\lambda_h} X_{lh}) \) emerges due to the definition of a per capita LES. As this third component takes a positive value, it works contrary to the disutility component and increases the marginal returns from migration. This latter component represents the second part of the feedback between migration and the consumption sphere.

The mechanics of Equation (4.9) are best illustrated through the effect a supposed increase in \( w_m \). First of all, a rising \( w_m \) requires that the right hand side of the equation increases, too. This raises the household shadow wage and the time allocated to migration. However, both movements are counteracted by an
increase in the term \((p_a X_{ah} + p_p X_{ph} + \frac{\psi}{X_h} X_{lh})\). This increase happens due to the higher shadow wage, a higher income and, as a consequence of the latter and a smaller household size, an increase in per capita consumption quantities (compare Equation (4.5)). Ultimately, a new equilibrium (which also involves second round effects through changes in income and quantities consumed) with a higher level of migration is established.

At this juncture the overall behavior of the equation and, in particular, the amount of labor shifted to migration as a response to the wage shock, hinges on the calibration of the parameters \(\lambda_h, \varepsilon_{T_{mh}}\) and \(\delta_{T_{mh}}\). First of all, the initial values of all terms other than the disutility component are determined \(a\ priori\) through the data used in the SAM which underlies each village equilibrium model. Furthermore, the initial amount of time \(T_{mh}\) dedicated to migration is also given. This implies that \(\lambda_h, \varepsilon_{T_{mh}}\) and \(\delta_{T_{mh}}\) have to be calibrated in a way that the value of the disutility component allows the equation to be true. In other words, the value of the disutility component must equal the difference between the marginal returns from migration and the shadow wage minus the value of per capita household consumption. In addition, the values of \(\lambda_h, \varepsilon_{T_{mh}}\) and \(\delta_{T_{mh}}\) determine how fast the disutility component changes from a change in migration, i.e. how much labor has to be shifted to migration to achieve a given change in marginal disutility. The less labor is necessary for a given change in disutility, the faster the equilibrium is established and the weaker is the migration response of the household. Consequently, Equation (4.9) represents a utility function based implementation of different migration responses of agricultural households, allowing to account for supply side factors in a theoretically consistent manner. It should, however, be emphasized again that the content of this paper represents a work in progress and at the time of paper submission, the functioning of the calibration (and consequently the entire model) in a computer based implementation was not attempted.

The second activity households are engaged in is local off-farm employment. As in the case of migration, the "output" of the local off-farm activity is the amount of labor supplied to the local off-farm labor market. Accordingly, the production function

\[
Q_{oh} = T_{oh} \quad \forall \ h \in H
\]  

(4.10)

maps the time worked in local-off farm jobs into an output variable \(Q_{oh}\). Accord-
ing to Equation (3.9), the factor demand equation for local off-farm labor must ensure that the wage earned equals the household shadow wage minus a utility component which reflects the disutility generated through the participation in the activity:

$$w_o = \frac{\psi}{\lambda_h} + \frac{1}{\lambda_h} \varepsilon_{T_o,h} \delta_{T_o,h} T_{oh}^{\delta_{T_o,h} - 1} \quad \forall \ h \in H$$

(4.11)

Once again, the disutility component drives a wedge between the shadow wage and the market wage rate. The points made above about the calibration of the parameters $\lambda_h$, $\varepsilon_{T_m,h}$ and $\delta_{T_m,h}$ apply. Just as in case of migration, it is possible to incorporate assumptions rooted in the utility concept which exert an influence on the strength of the households’ responses from changes in the market wage rate.

The final income generating activity of the households is agricultural production. As mentioned above, the agricultural good is produced with a Cobb-Douglas production technology

$$Q_{ah} = \alpha_h T_{fh}^{\beta_{T,h}} V_h^{\beta_{v,h}} A_h^{\beta_{A,h}} \quad \forall \ h \in H.$$  

(4.12)

In this equation, the $\alpha$ is an efficiency parameter while the parameters $\beta$ represent the cost shares of the respective inputs. The functions for factor demand from agriculture are

$$V_h = \frac{\beta_v Q_{a,P_a}}{v} \quad \forall \ h \in H$$

(4.13)

for the variable input and

$$A_h = \frac{\beta_{A,h} Q_{a,P_a}}{a} \quad \forall \ h \in H$$

(4.14)

for land. While these equations correspond to standard factor demand functions for a Cobb-Douglas production technology, the inclusion of farm labor into the utility function leads to a modified version of the demand for this factor in the agricultural activity:

$$T_{fh} = \frac{\beta_{T,h} Q_{a,P_a}}{1 / \lambda_h \varepsilon_{T,h} \delta_{T,h} T_{fh}^{\delta_{T,h} - 1} + \psi / \lambda_h} \quad \forall \ h \in H.$$  

(4.15)

Equation 4.15 states that the marginal value product of family labor ($MVP_l$) in
agriculture has to be equal to the cost of the factor, which consists of two components. These components, again, are the shadow wage, i.e. the opportunity cost of labor, and a utility component. Just as in migration and the off-farm activity it allows the MVP and the shadow wage to differ by a margin which corresponds to the disutility generated through the participation in the activity.

The points made above regarding the calibration of the parameters \( \lambda_h, \varepsilon_{bh} \) and \( \delta_{T_{fh}} \) remain valid.

Equations (4.7) and (4.9) through (4.15) build the production block of the model which includes the allocation of family labor to migration and other activities. Now it is possible to define the remaining part of the model, including total household income as the sum of the returns from the factors owned by the households

\[
I_h^T = \left( \frac{\psi}{\lambda_h} + \frac{1}{\lambda_h} \varepsilon_{T_{fh}} \delta_{T_{fh}} \sigma_{T_{fh}} \right) T_{fh} + w_o T_{oh} + \kappa w_m T_{mh} + \frac{\psi}{\lambda_h} X_{T_{lh}} + aA_{fh} \quad \forall \ h \in H.
\]

The first term on the right hand side of Equation (4.16) consists of the returns from household labor employed in agriculture. For an applied model, two possibilities exist to deal with this term. First, one may evaluate \( \frac{\psi}{\lambda_h} \) with an estimated value (Jacoby, 1993) and assume that possible residual profits which remain after deducting all payments to farm inputs and factors (which include household labor evaluated at the shadow wage) from gross revenues represent the disutility component. Alternatively, the assumption can be made that the disutility from working on the farm experienced by the households equals zero.

In this case, factor remuneration and, implicitly, the household shadow wage equal gross revenues minus the payments to purchased inputs and land. The second and third term on the right hand side represent household income from local off-farm employment and remittances. Please note that similar to the returns from farm labor, both terms can be decomposed into payments to labor evaluated at the household shadow wage and a disutility component. The first term in the second line of the equation counts the value of leisure as income. This is necessary because leisure is included into the model as a consumption good (see equation (4.4)), which implicitly means that households purchase amounts of leisure they consume themselves. Finally, households receive income from land according to their endowment. This income is taken into account by the last term in the equation.
The expenditure and production blocks defined in the preceding paragraphs by Equations (4.2) - (4.5), (4.7) and (4.9) through (4.15) along with the income equations (4.6) and (4.16) describe the behavior of the households in the model. To complete the model, it is necessary to define additional household and village level constraints. The equilibrium conditions incorporate the tradability assumptions made above and determine the price formation mechanisms for the different goods and factors. Thus, they define the market environment within which the households operate.

The first set of equilibrium conditions involves the household level constraints. In case of the produced agricultural good, households have the possibility to either consume or sell it to the market. Due to the lack of other uses, total consumption and sales, denoted as $S_{ah}$, must be equal:\textsuperscript{16}

$$Q_{ah} = \left(N_h - \frac{T_{mh}}{E} + \Delta D_h\right)X_{ah} + S_{ah} \quad \forall \ h \in H$$ (4.17)

The manufactured good and the variable input, which households are not endowed with and which cannot be produced by themselves, have to be entirely purchased from the market. Accordingly,

$$\left(N_h - \frac{T_{mh}}{E} + \Delta D_h\right)X_{ph} = P_{ph} \quad \forall \ h \in H$$ (4.18)

and

$$V_h = P_{Vh} \quad \forall \ h \in H$$ (4.19)

form the respective household level commodity balances.

In case of land, rentals by households from and to the village market are possible. This implies that the sum of the use of land by the agricultural activity and net rentals $R_h$ must equal the land endowment of each household:

$$\overline{A}_h = A_h + R_h \quad \forall \ h \in H$$ (4.20)

From Equation (4.20) net land rentals are defined as the residual from household land endowments and land use by the specific household.

The last commodity balance at the household level is a time constraint. As

\textsuperscript{16}This involves the assumption that households are always net-sellers of the agricultural good which can be maintained for marginal changes in exogenous variables.
labor time cannot be purchased, total time use by the households for productive activities and leisure must equal the time endowment:

\[ T_h = T_{fh} + T_{oh} + T_{mh} + X_{lh} \left( N_h - \frac{T_{mh}}{E} \right) \quad \forall \ h \in H \]  

(4.21)

In addition, and although not strictly a commodity balance but a balance of payments, households cannot spend more income than they earn:

\[ I_T^h = \left( N_h - \frac{T_{mh}}{E} + \Delta D_h \right) \left( p_a X_{ah} + p_p X_{ph} + \frac{\psi}{\lambda_h} X_{lh} \right) \]  

(4.22)

Note that unlike the budget constraint of the theoretical model, leisure is now explicitly included. Furthermore, (net) expenses on variable inputs and land rentals are not included because the income equation (4.16) states net household income.\(^{17}\)

Following the household commodity balances and balance of payments, village level balances constitute the second set of system constraints in the model and define the village equilibrium framework. For household labor, which is not traded between households and which cannot be purchased, it is sufficient to formulate balances at the household level. Hence, no additional equation has to be added. In case of the agricultural good, total supply in the village must equal total demand

\[ \sum_{h \in H} Q_{ah} = \sum_{h \in H} \left( N_h - \frac{T_{mh}}{E} + \Delta D_h \right) X_{ah} + \sum_{h \in H} S_{ah} \]  

(4.23)

For the manufactured good and the variable input, total demand must equal total imports\(^{18}\)

\[ \sum_{h \in H} P_{ph} = \sum_{h \in H} \left( N_h - \frac{T_{mh}}{E} + \Delta D_h \right) X_{ph} \]  

(4.24)

and

\[ \sum_{h \in H} P_{Vh} = \sum_{h \in H} V_h. \]  

(4.25)

\(^{17}\)That is, instead of gross revenues only factor returns are included.

\(^{18}\)It should be noted that Equations (4.23) - (4.25) are already fulfilled by household level balances and thus could be excluded from of an applied model. Nevertheless, they are included here to provide a complete representation of the village equilibrium framework.
For land, a village market exists. Consequently, total demand for land in the village must equal total supply, which is the village’s land endowment:

$$\sum_{h \in H} A_h = \sum_{h \in H} \bar{A}_h$$  \hspace{1cm} (4.26)

Similar to the household level, a village balance of payments has to be defined, stating that total payments on imports must correspond to the inflow of payments to the village:

$$p_p \sum_{h \in H} \left( N_h - \frac{T_{mh}}{E} + \Delta D_h \right) X_{ph} + v V_h$$  

$$= p_a \sum_{h \in H} S_{ah} + w_o \sum_{h \in H} T_{oh} + \sum_{h \in H} \left( \kappa_h w_m Q_{mh} \right)$$  \hspace{1cm} (4.27)

The final set of equations of the village equilibrium model concerns the relevant decision making prices. In this version of the model, it is sufficient to assign an exogenously fixed value to prices for which the village is a price taker, i.e. the prices of village tradables and off-farm labor including migration

$$p_a = \bar{p}_a$$  \hspace{1cm} (4.28)

$$p_p = \bar{p}_p$$  \hspace{1cm} (4.29)

$$v = \bar{v}$$  \hspace{1cm} (4.30)

$$w_o = \bar{w}_o$$  \hspace{1cm} (4.31)

$$w_m = \bar{w}_m$$  \hspace{1cm} (4.32)

where a bar marks exogenously fixed values.

The rental price for land $a$ is determined by the interplay between supply and demand within the village and the household shadow price $\frac{\psi}{\lambda h}$ is formed inside each household. Hence, the only equation which has to be added here is

$$p_{lh} = \frac{\psi}{\lambda h}$$  \hspace{1cm} (4.34)

which defines the price of leisure used in the sum notations of the demand equations (4.2) - (4.4) and completes the model.
5 Conclusions and outlook

The stylized village equilibrium model elaborated in Section 4 proposes a utility function based approach towards modeling migration, taking into account feedbacks between migration and the consumption sphere of a household. This approach offers the possibility to incorporate household preferences with respect to migration into the model and to allow for differing responses from changes in incentives to migrate. Offering a supply side perspective rooted in a theoretical agricultural household model goes beyond the demand side oriented modeling encountered in current village equilibrium approaches.

In the depiction of labor allocation, including migration, the adequate calibration of the factor demand functions, which ultimately govern the labor market participation of the household, plays a crucial role. Evidently, as the parameters have no direct real-world equivalent which would allow for an empirical estimation, the calibration involves a search procedure to find parameter values which yield labor supply responses of households which might be considered realistic by the modeler. Most ideally, however, the modeling exercise would be accompanied by econometric work to analyze household labor supply responses to changes in relative wage rates (see, for example, Skoufias, 1994; Sicular and Zhao, 2004). The parameters in the factor demand functions would be calibrated to match the findings of this empirical work. This approach offers the perspective of obtaining results which have a better empirical foundation.

An important feature of the model worth emphasizing, is that households are able to be engaged in several income generating activities simultaneously. This happens without the necessity of reverting to analytical constructs, such as the introduction of profits into off-farm activities or the imposition of quantitative restrictions on off-farm employment. Instead, an equilibrium situation is achieved in which households allocate certain amounts of labor to different activities. This equilibrium depends on the preferences of households, on the one hand, and on the market environment – demand for labor reflected in a particular wage rate – on the other hand. The focus in this study was to model migration while allowing for migration responses which are able to differ between households. The approach presented above, however, could be applied to a larger variety of problems. More generally, it is possible to model diversification of income sources while avoiding complete specialization – something which is often found in reality. More specifically, a more sophisticated applica-
tion of the approach could be used to model regional labor market equilibria in national or global CGE models. Through the inclusion of households’ geographical preferences for employment, a criterion for employment choices, such as limited spatial mobility, which the household considers in addition to the wage rate offered on the market could be introduced. Nevertheless, for a moment the village equilibrium model presented here offers a sound alternative to current village equilibrium models as it is able to capture household preferences better.

References


A Lagrangian of the household problem and first order conditions

The Lagrangian expression associated with the household problem laid out in Section 3 is:

\[
\mathcal{L} = U^C(X_a, X_p, X_t) + U^L(T_f, T_o, T_m) + \\
+ \lambda[p_a f(T_f, V, A) + w_o T_o + r(T_m; w_m) + a(A - A) - \\
- (N - \frac{T_m}{E} + \Delta D)(p_a X_a + p_p X_p - vV) + \\
+ \psi[T - T_f - T_o - T_m - X_t(N - \frac{T_m}{E})].
\]  

(A.1)
Partial differentiation of the Lagrangian leads to the following first order conditions:

\[
\frac{\partial L}{\partial X_a} = \frac{\partial U^C}{\partial X_a} - \lambda_p (N - \frac{T_m}{E} + \Delta D) = 0 \quad (A.2)
\]

\[
\frac{\partial L}{\partial X_p} = \frac{\partial U^C}{\partial X_p} - \lambda_p (N - \frac{T_m}{E} + \Delta D) = 0 \quad (A.3)
\]

\[
\frac{\partial L}{\partial X_l} = \frac{\partial U^C}{\partial X_l} - \psi (N - \frac{T_m}{E}) = 0 \quad (A.4)
\]

\[
\frac{\partial L}{\partial T_f} = \frac{\partial U^L}{\partial T_f} + \lambda_p \frac{\partial f(\cdot)}{\partial T_f} - \psi = 0 \quad (A.5)
\]

\[
\frac{\partial L}{\partial T_o} = \frac{\partial U^L}{\partial T_o} + \lambda_w - \psi = 0 \quad (A.6)
\]

\[
\frac{\partial L}{\partial T_m} = \frac{\partial U^L}{\partial T_m} + \lambda \frac{\partial r(\cdot)}{\partial T_m} + \lambda_p X_a + p_X \psi - \psi = 0 \quad (A.7)
\]

\[
\frac{\partial L}{\partial A} = \lambda_p \frac{\partial f(\cdot)}{\partial A} - \lambda a = 0 \quad (A.8)
\]

\[
\frac{\partial L}{\partial V} = \lambda_p \frac{\partial f(\cdot)}{\partial V} - \lambda v = 0 \quad (A.9)
\]
\[ \frac{\partial L}{\partial \lambda} = p_a f(T_f, V, A) + w_o T_o + r(T_m; w_m) + a(\bar{A} - A) - \\
- (N - \frac{T_m}{E} + \Delta D)(p_a X_a + p_p X_p) - vV = 0 \quad (A.10) \]

\[ \frac{\partial L}{\partial \psi} = T - T_f - T_o - T_m - X_i (N - \frac{T_m}{E}) = 0 \quad (A.11) \]